$(\mathcal{F}, h)$ cone upper class on Fixed Point Results in Quasi–Cone Metric Space for Generalized Contractive Mappings Using Diameter of Orbits

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Abstract

In this paper is defined cone upper class. Some fixed point theorems related to (F, h) - cone upper class in quasi - cone metric space are given. As applications to illustrate these results are provided some examples.

1. Introduction

In 2007, Huang and Zhang [13] introduced the concept of cone metric space replacing the real axis in distance function by an ordered Banach space. They proved some fixed point theorems generalizing the result of Kannan [11], Chatterja [12]. There are many authors like Binayak et al. [10], Raja and Vaezpour [14], Ding [18] have studied fixed point in cone metric spaces.

Later, in 2009 Abdeljawad and Karapinar [15] generalized cone metric spaces into quasi cone metric spaces. Using this generalization many authors like Sila et.al [16], Shaddad and Noorani [7], ect have given their contribution fixed points for contractions given in [8].

In 2012, Karapinar and Samet [1] proved some fixed point theorems related $\alpha - \psi$ contractive mappings in metric space and then Bilgili et al. [2] extended these result to quasi-cone metric space.

Later, Sila et al. [17] generalized these results in quasi-cone metric space using the diameter of orbits.

Recently A. H. Ansari [21] introduced the concept of (F, h) upper class of type I in metric spaces. In this paper is defined (F, h) upper class of type I and a class of $\alpha - \mu - \psi$ contractive mappings in quasi cone metric space and is proved some fixed point theorems which generalize the results of [2], [17].

Below there are given some preliminaries that are used for proving the new results.

Definition 1. 1 *[13] Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:*

1. P is closed, $P \neq \emptyset, P \neq \{0\}$;

2. For all $x, y \in P, \alpha x + \beta y \in P$, where $\alpha, \beta \in \mathbb{R}^+$;

3. if $x \in P$ and $-x \in P$ implies $x = 0$.

For a given cone $P \subset E$, we can define a partial ordering "$\leq$ " with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}\, P$, where $\text{int}\, P$ denotes the interior of P.

The cone P is called normal if there is a number $k > 0$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$, for all $x, y \in P$. The least positive k satisfying this, is called the normal constant of P.

Definition 1.2 *[13] Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies following conditions:*

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

2. $d(x, y) = d(y, x)$ for all $x, y \in X$;

3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Definition 1.3 *[15] Let X be a nonempty set. Suppose the mapping $q: X \times X \longrightarrow P$ satisfies following conditions:*

1. $0 \leq q(x, y)$ for all $x, y \in X$;

2. $q(x, y) = 0$ if and only if $x = y$;

3. $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

Then, q is called a *quasi–cone metric* on X and (X, q) is called a *quasi–cone metric space*.

Remark 1.4 *Note that any cone metric space is a quasi–cone metric space.*

Shaddad and Noorani [7] introduced the appropriate generalization in quasi–cone metric spaces by considering the established notions in quasi–metric spaces.

Definition 1. 5 [7] *Let (X, q) be a quasi–cone metric space.*

A sequence $\{x_n\}$ in X is called right (left) Cauchy if for each $c \in \operatorname{int} P$, there is $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) \ll c$ ($q(x_m, x_n) \ll c$ resp.) for all $n \geq m \geq n_0$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called Cauchy if and only if it is both left and right Cauchy.

Definition 1. 6 [7] *Let (X, q) be a quasi–cone metric space.* Let $\{x_n\}_{n\in\mathbb{N}}$ in X. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is right convergent to $x \in X$ if $q(x, x_n) \to 0$. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x \in X$ if the sequence is right and left convergent to x. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 1. 7 [7] *A quasi–cone metric space (X, q) is called complete if every Cauchy sequence in X converges.*

Definition 1. 8 [10] *Let $O(x) = \{x, Tx, T^2x, \ldots\}$ where $x \in X$. The set $O(x)$ is called orbit of x.*

Using the same method as in [5], there is defined diameter of a set in quasi - cone metric space.

Definition 1. 9 ([5], [9]) *Let $M \subseteq X$, where X is a quasi-cone metric space.*

$\delta(M) = \sup\{q(x, y), q(y, x), x, y \in M\}$ *is called diameter of M.*

Define $\delta(O(x) \cup O(y)) = \max\{q(T^i x, T^j x), q(T^k y, T^m y), q(T^i x, T^k y)\}$ for $i, j, k, m \in \mathbb{N}_0$.

The orbit $O(x)$ is called bounded if there exist a $c \in P$, $\delta(O(x)) \leq c$.

Definition 1. 10 [4]*The continuous function $\psi: P \to P$ which satisfies the following conditions:*

1. $\forall t \in P, \psi(t) < t$;
2. $\forall t_1, t_2 \in P$, for $t_1 < t_2$ implies $\psi(t_1) < \psi(t_2)$;
3. $\lim_{n\to\infty} \|\psi^n(c)\| = 0$

is called a *comparison function in cone P.*

Definition 1. 11 [3] *Let (X, q) be a quasi--cone metric space and $T: X \to X$ be a given function.*

The map T is $\alpha - \psi$ contractive mapping if there exist two functions $\alpha: X \times X \to [0, +\infty)$ and ψ a comparison function which satisfy the nonlinear contraction condition:

$$\alpha(x, y) q(T(x), T(y)) \leq \psi(q(x, y)).$$

Definition 1. 12 [23] *Let $T: X \to X$ and $\alpha: X \times X \to [0, +\infty)$. The mapping T is α −admissible if for all $x, y \in X$ the following implication is true*

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Definition 1. 13 *[6] Let* $T: X \to X$ *and* $\mu: X \times X \to [0, +\infty)$. *The mapping* T *is* $\mu - subadmissible$ *if for all* $x, y \in X$ *and* $\mu(x, y) \leq 1$ *the inequality* $\mu(Tx, Ty) \leq 1$ *holds*.

Definition 1. 14 *[19, 20]A mapping* $F: P^2 \to P$ *is called cone* C*-class function if it is continuous and satisfies following axioms:*

(1) $F(s, t) \leq s$;

(2) $F(s, t) = s$ *implies that either* $s = \theta$ *or* $t = \theta$; *for all* $s, t \in P$.

We denote C - class functions as $\mathcal{C}$.

In 2014 the concept of pair $(\mathcal{F}, h)$ is an upper class was introduced by A. H. Ansari in [21]

Definition 1. 15 *[21, 22] The function* $h: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a function of subclass of type I, if* $x \geq 1 \Rightarrow h(1, y) \leq h(x, y)$ *for all* $y \in \mathbb{R}^+$.

Definition 1. 16 *[21, 22] Let* $h, \mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, *then we say that the pair* $(\mathcal{F}, h)$ *is an upper class of type I, if* h *is a function of subclass of type I and:*

(i) $0 \leq s \leq 1 \Rightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$,

(ii) $h(1, y) \leq \mathcal{F}(s, t) \Rightarrow y \leq st$ *for all* $t, y \in \mathbb{R}^+$.

2. Main Results

Definition 2. 1 *The function* $h: \mathbb{R}^+ \times \mathbb{P} \to \mathbb{P}$ *is a cone function of subclass of type I, if* $x \geq 1 \Rightarrow h(1, y) \leq h(x, y)$ *for all* $y \in$ P.

Example 2. 2 *Define* $h: \mathbb{R}^+ \times \mathbb{P} \to \mathbb{P}$, *where* $P = \{(m, n) \in R^2, m, n \geq 0\}$, *by:*

1. $h(x, (m, n)) = x(m, n)$;
2. $h(x, (m, n)) = e^x(m, n)$,
3. $h(x, (s, t)) = (s, t)$;
4. $h(x, (s, t)) = \frac{1}{n+1}(\sum_{i=0}^{n} x^i)(s, t), n \in \mathbb{N}$;

for all $x \in \mathbb{R}^+$. Then h is a cone function of subclass of type I.

Definition 2. 3 *Let* $h, \mathcal{F}: \mathbb{R}^+ \times \mathbb{P} \to \mathbb{P}$. *The pair* $(\mathcal{F}, h)$ *is an upper class of type I, if* h *is a cone function of subclass of type I and:*

(i) $for\ 0 \leq s \leq 1\ implies\ \mathcal{F}(s,t) \leq \mathcal{F}(1,t)$,
(ii) $if\ h(1,y) \leq \mathcal{F}(s,t)\ then\ y \leq st\ for\ all\ t, y \in \mathbb{R}^+$.

Example 2. 4 *Define* $h, \mathcal{F}: \mathbb{R}^+ \times \mathbb{P} \to \mathbb{P}$ *by:*

1. $h(x,(m,n)) = x(m,n)$ and $\mathcal{F}(s,(u,w)) = s(u,w)$;

2. $h(x,y) = \frac{1}{n+1}(\sum_{i=0}^{n} x^i)y$, $n \in \mathbb{N}$ and $\mathcal{F}(s,t) = st$;

where $P = \{(m,n) \in R^2, m, n \geq 0\}$. Then the pair $(\mathcal{F}, h)$ is an cone upper class of type I.

Theorem 2. 5 *Let* (X,q) *be a complete Hausdorff quasi-cone metric space and let* $T: X \to X$ *be a continuous function that satisfies the nonlinear contraction condition:*

$$h(\alpha(x,y), q(T(x),T(y))) \leq \mathcal{F}(\mu(x,y), \psi(M(x,y))).$$

$M(x,y) = \max\{q(x,y), q(Tx,x), q(Ty,y), \frac{1}{2}[q(Tx,y) + q(x,Ty)]\}$ for all $x, y \in X$, the pair $(\mathcal{F}, h)$ is an cone upper class of type I, and $\psi: P \to P$ is a continuous function that for every $t \in P, \psi(t) < t$ and $\forall t_1, t_2 \in P$, for $t_1 < t_2$ implies $\psi(t_1) < \psi(t_2)$.

Suppose that

1. $\alpha: X \times X \to [0, \infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1, \mu(T^n x_0, T^m x_0) \leq 1$ for every $n, m \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is convergent to x^*.

Proof. Define the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$. The sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is right Cauchy.

Indeed, firstly, is proved that the sequence $\{q(T^{n+1}x_0, T^n x_0)\}_{n \in \mathbb{N}}$ is monotonic decreasing as follows:

$$\begin{gathered} h(1, q(T^{n+1}x_0, T^n x_0)) \leq h(\alpha(T^n x_0, T^{n-1}x_0), q(T^{n+1}x_0, T^n x_0)) \\ \leq \mathcal{F}(\mu(T^n x_0, T^{n-1}x_0), \psi(M(T^n x_0, T^{n-1}x_0))) \\ \leq \mathcal{F}(1, \psi(M(T^n x_0, T^{n-1}x_0))) \\ q(T^{n+1}x_0, T^n x_0) \leq \psi(M(T^n x_0, T^{n-1}x_0)) \end{gathered}$$

where

$$M(T^n x_0, T^{n-1} x_0) = \left\{ q(T^n x_0, T^{n-1} x_0), q(T^{n+1} x_0, T^n x_0), q(T^n x_0, T^{n-1} x_0), \frac{1}{2} [q(T^{n+1} x_0, T^{n-1} x_0) + q(T^n x_0, T^n x_0)] \right\} = \max\{q(T^n x_0, T^{n-1} x_0), q(T^{n+1} x_0, T^n x_0)\}$$

Case 1. $M(T^n x_0, T^{n-1} x_0) = q(T^{n+1} x_0, T^n x_0)$, so $q(T^{n+1} x_0, T^n x_0) \leq \psi(q(T^{n+1} x_0, T^n x_0))$ which is a contradiction.

Case 2. $M(T^n x_0, T^{n-1} x_0) = q(T^n x_0, T^{n-1} x_0)$, so

$$q(T^{n+1} x_0, T^n x_0) \leq \psi(q(T^n x_0, T^{n-1} x_0)) \leq q(T^n x_0, T^{n-1} x_0) \text{ for all } n \geq 1.$$

Let $C_k = \sup\{q(T^i x_0, T^j x_0), i > j > k\}$. The sequence $\{C_k\}$ is monotonic decreasing and lower bound, so it converges to $C_0 \in P$. As a result, it yields

$$\forall p \in \mathbb{N}, i_p > j_p > p, C_p - \frac{C_0}{p} \leq q\left(T^{i_p} x_0, T^{j_p} x_0\right) \leq C_p \textit{ implies } q(T^{i_p} x_0, T^{j_p} x_0) \to C_0 \text{ as } p \to \infty$$

Below is proved that $C_0 = 0$.

$$q(T^{i_p+1} x_0, T^{j_p+1} x_0) \leq \alpha(T^{i_p} x_0, T^{j_p} x_0) q(T^{i_p+1} x_0, T^{j_p+1} x_0) \leq \psi(M(T^{i_p} x_0, T^{j_p} x_0))$$

Taking the limit when $p \to +\infty$, is taken $C_0 \leq \psi(C_0)$, so $C_0 = 0$.

Using the same method, it can be proved that the sequence $\{T^n x_0\}$ is left Cauchy. So it is a Cauchy sequence and since the space is complete, it is convergent to x^*.

$$\lim_{n\to\infty} q(T^n x_0, x^*) = \lim_{n\to\infty} q(x^*, T^n x_0) = 0.$$

Since T is continuous $\lim_{n\to\infty} q(T^n x_0, T x^*) = \lim_{n\to\infty} q(T(T^{n-1} x_0), T x^*)) = 0$.

Furthermore, $\lim_{n\to\infty} q(T x^*, T^n x_0) = \lim_{n\to\infty} q(T x^*, T(T^{n-1} x_0)) = 0$.

Thus, $\lim_{n\to\infty} q(T^n x_0, T x^*) = \lim_{n\to\infty} q(T x^*, T^n x_0) = 0$.

As X is Hausdorff, it is true that $x^* = T x^*$ and x^* is a fixed point of T.

In the following theorem the function $T: X \to X$ is non-continuous.

Theorem 2. 6 *Let (X, q) be a complete, Hausdorff quasi–cone metric space where cone is normal with constant of normality K and $T: X \to X$ be a function that satisfies the nonlinear contraction condition:*

$$h(\alpha(x,y), q(Tx,Ty))) \leq \mathcal{F}(\mu(x,y), \psi(N(x,y))$$

where $N(x,y)) = \max\{q(x,y), \frac{1}{a}[q(Tx,x) + q(Ty,y)], \frac{1}{a}[q(Tx,y) + q(x,Ty)]\}$ for all $x, y \in X, a \geq K, \psi: P \to P$ is a continuous function that for every $t \in P, \psi(t) < t$ and $\forall t_1, t_2 \in P$, for $t_1 < t_2$ implies $\psi(t_1) < \psi(t_2)$.

Suppose that

1. $\alpha: X \times X \to [0, +\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1, \mu(T^n x_0, T^m x_0) \leq 1$, for every $n, m \in \mathbb{N}$.

2. If $\{T^n x\}$ is a sequence such that $\alpha(T^{n+1}x_0, T^n x_0) \geq 1, \mu(T^{n+1}x_0, T^n x_0) \leq 1$ for all n and $T^n x \to x^*$ as $n \to \infty$, then there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$ such that $\alpha(T^{n_k}x, x^*) \geq 1, \alpha(x^*, T^{n_k}x) \geq 1, \mu(T^{n_k}x, x^*) \leq 1, \mu(x^*, T^{n_k}x) \leq 1$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Define the sequence $\{T^n x_0)\}_{n\in\mathbb{N}}$. Continuing the same procedure as Theorem 2. 5, we prove that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is Cauchy and so it converges to x^*.

From Condition 2 of theorem, there exist a subsequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$, such that $\alpha(T^{n_k}x, x^*) \geq 1$, $\alpha(x^*, T^{n_k}x) \geq 1$ for all $k \in N$.

$$\begin{aligned} h(1, q(Tx^*, T^{n_k+1}x_0)) &\leq h(\alpha(x^*, T^{n_k}x_0), q(Tx^*, T^{n_k+1}x_0)) \\ &\leq \mathcal{F}(\mu(x^*, T^{n_k}x_0), \psi(N(Tx^*, T^{n_k+1}x_0))) \\ &\leq \mathcal{F}(1, \psi(N(Tx^*, T^{n_k+1}x_0))) \end{aligned}$$

$$\begin{aligned} q(Tx^*, T^{n_k+1}x_0) &\leq \psi(N(Tx^*, T^{n_k+1}x_0) < N(Tx^*, T^{n_k}x_0) \\ &= \max\{q(x^*, T^{n_k}x_0), \frac{1}{a}[q(Tx^*, x^*) + q(T^{n_k+1}x_0, T^{n_k}x_0)], \\ &\frac{1}{a}[q(Tx^*, T^{n_k}x_0) + q(T^{n_k+1}x_0, x^*)]\} \end{aligned}$$

Taking the limit when $n_k \to +\infty$, $N(Tx^*, T^{n_k}x_0)$ converges to $\frac{1}{a}q(Tx^*, x^*)$.

From $q(Tx^*, T^{n_k+1}x_0) \leq \psi(N(Tx^*, T^{n_k}x_0)) < N(Tx^*, T^{n_k}x_0)$ and taking the limit when $n_k \to +\infty$, is taken the following inequality

$$q(Tx^*, x^*) < \frac{1}{a}q(Tx^*, x^*),$$

which is a contradiction. Consequently, $q(Tx^*, x^*) = 0$ and $Tx^* = x^*$ and x^* is a fixed point of T.

In the following theorem the function T is taken non-continuous.

Theorem 2. 7 *Let* (X,q) *be a complete Hausdorff quasi–cone metric space and let* $T: X \to X$ *be a continuous function that satisfies the nonlinear contraction condition:*

$$h(\alpha(x,y)q(T(x),T(y))) \leq \mathcal{F}(\mu(x,y),\psi(\delta(O(x)\cup O(y))))$$

for all $x,y \in X$, where $\psi: P \to P$ is a comparison function and the pair $(\mathcal{F},h)$ is a cone upper class of type *I*. Suppose that $\alpha,\mu: X\times X \to [0,+\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$ and $\mu(T^n x_0, T^m x_0) \leq 1$ for every $n,m \in N$.

Moreover for $x_0 \in X$, the orbit $O(x_0)$ is bounded.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n\in N}$ is convergent to x^*.

Proof. From condition of theorem, there exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$ and $\mu(T^n x_0, T^m x_0) \leq 1$ for every $n,m \in \mathbb{N}$. Define the sequence $\{T^n x_0\}_{n\in N}$.

Below is proved that the sequence $\{T^n x_0\}_{n\in N}$ is right Cauchy.

Taking $x = T^{n+i}x_0, y = T^{n+j}x_0$, where $i,j,n \in N$ and $i > j$.

$$\begin{aligned} h(1,q(Tx,Ty)) &\leq h(\alpha(x,y),q(Tx,Ty)) \\ &\leq \mathcal{F}(\mu(x,y),\psi(\delta(O(x)\cup O(y)))) \\ &\leq \mathcal{F}(1,\psi(\delta(O(x)\cup O(y)))) \end{aligned}$$

As a result

$$\begin{aligned} q(Tx,Ty) &= q(T^{n+i+1}x_0, T^{n+j+1}x_0) \\ &\leq \psi(\delta(O(x)\cup O(y))) \\ &= \psi(\delta(O(T^{n+i}x_0)\cup O(T^{n+j}x_0))) \\ &\leq \psi(\delta(O(T^n x_0)) \end{aligned}$$

So, it is true that $(T^{n+i+1}x_0, T^{n+j+1}x_0) \leq \psi(\delta(O(T^n x_0))$

for every $i,j,n \in N$ and $i > j$.

Furthermore

$$\begin{aligned} q(T^{n+i+1}x_0, T^{n+j+1}x_0) &\leq \delta(O(T^{n+1}x_0)) \\ &= \max\{q(T^{n+i+1}x_0, T^{n+j+1}x_0), i,j \in \mathbb{N}\} \\ &\leq \psi(\delta(O(T^n x_0))). \end{aligned}$$

From this, it yields

$$\begin{aligned} q(T^{n+i+1}x_0, T^{n+j+1}x_0) &\leq \psi(\delta(O(T^n x_0))) \\ &\leq \psi^2(\delta(O(T^{n-1}x_0))) \leq \cdots \\ &\leq \psi^n(\delta(O(x_0))) \leq \psi^n(c) \end{aligned}$$

Due to $\lim_{n\to\infty} \|\psi^n(c)\| = 0 \Leftrightarrow (\forall \frac{\varepsilon}{K} > 0)(\exists n_0 \in N)(\forall n > n_0 \Rightarrow \|\psi^n(c)\| < \frac{\varepsilon}{K})$, where K is the constant of normality of cone, we have

$$\left\|q(T^{n+i+1}x_0, T^{n+j+1}x_0)\right\| \leq K\|\psi^n(c)\| < K \cdot \frac{\varepsilon}{K} = \varepsilon, \text{ for } n > n_0, \text{ and } i > j.$$

Consequently, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is right Cauchy. In the same manner it can be proved that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is left Cauchy. Since (X, q) is complete, there exists a point $x^* \in X$ such that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to $x^* \in X$, so $q(T^n x_0, x^*) \to 0$ as $n \to \infty$ and $q(x^*, T^n x_0) \to 0$ as $n \to +\infty$.

The point x^* is a fixed point of T, $Tx^* = x^*$. Indeed, it is true that $\lim_{n\to+\infty} q(T^n x_0, x^*) = 0$ and $\lim_{n\to+\infty} q(x^*, T^n x_0) = 0$. By using the continuity of T, we have $\lim_{n\to+\infty} q(T(T^n x_0), Tx^*) = 0$ and $\lim_{n\to+\infty} q(Tx^*, T(T^n x_0)) = 0$. By uniqueness of the limit, it is concluded that $Tx^* = x^*$.

Theorem 2. 8 *Let (X, q) be a complete Hausdorff quasi–cone metric space and let $T: X \to X$ be a function that satisfies the nonlinear contraction condition:*

$$h(\alpha(x,y)q(T(x),T(y))) \leq \mathcal{F}(\mu(x,y), \psi(\frac{1}{a}\delta(O(x) \cup O(y))))$$

for $x, y \in X$ and $a \geq K$, where $\psi: P \to P$ is a comparison function. Suppose that

1. $\alpha, \mu: X \times X \to [0, \infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$ and $\mu(T^n x_0, T^m x_0) \leq 1$ for every $n, m \in \mathbb{N}$.

2. If $\{T^n x\}_{n\in\mathbb{N}}$ is a sequence such that for $T^n x \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n_k} x\}$ of $\{T^n x\}_{n\in\mathbb{N}}$ such that $\alpha(T^{n_k}x, T^q x^*) \geq 1$, $\alpha(T^{n_k}x, x^*) \geq 1, \alpha(x^*, T^{n_k}x) \geq 1$ and $\mu(T^{n_k}x, T^q x^*) \leq 1, \mu(T^{n_k}x, x^*) \leq 1, \mu(x^*, T^{n_k}x) \leq 1$.

Moreover for every $z \in X$, the orbit $O(z)$ is bounded. Then T has a fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$. Define the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$. Using the same method as in Theorem 2. 7 it can be proved that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is Cauchy. Since (X, q) is complete, there exists a point $x^* \in X$ such that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to $x^* \in X$, so $q(T^n x_0, x^*) \to 0$ as $n \to +\infty$ and $q(x^*, T^n x_0) \to 0$ as $n \to +\infty$.

The next step is to prove that x^* is a fixed point of T, $Tx^* = x^*$. For this, it is needed to prove that the sequence $\{T^m x_0\}_{m\in\mathbb{N}}$ converges to x^*. Suppose that this sequence converges to $l \in X$.

$$h(1, q(T^{n_k+1}x^*, T^{n_k+1}x_0)) \leq h(\alpha(T^{n_k}x^*, T^{n_k}x_0), q(T^{n_k+1}x^*, T^{n_k+1}x_0))$$
$$\leq \mathcal{F}(\mu(T^{n_k}x^*, T^{n_k}x_0), \psi(\frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(T^{n_k}x_0)))$$
$$\leq \mathcal{F}(1, \psi(\frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(T^{n_k}x_0))))$$

So, the following inequality is true.

$$q(T^{n_k+1}x^*, T^{n_k+1}x_0) \leq \psi(\frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(T^{n_k}x_0)))$$
$$\leq \frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(T^{n_k}x_0))$$
$$= \frac{1}{a}\max\{q^{n_k+i}x^*, T^{n_k+j}x^*), q(T^{n_k+p}x_0, T^{n_k+r}x_0), q(T^{n_k+i}x^*, T^{n_k+p}x_0)\}$$

for $i, j, p, r \in \mathbb{N}_0$.

Taking the limit of both sides when $k \to \infty$, $q(l, x^*) < \frac{1}{a}q(l, x^*)$.

So this is a contradiction, and it yields $q(l, x^*) = 0$, $l = x^*$.

Now we prove that x^* is a fixed point of T.

$$h(1, q(T^{n_k+1}x^*, Tx^*)) \leq h(\alpha(T^{n_k}x^*, Tx^*), q(T^{n_k+1}x^*, Tx^*))$$
$$\leq \mathcal{F}(\mu(T^{n_k}x^*, Tx^*), (\frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(x^*))))$$
$$\leq \mathcal{F}(1, (\frac{1}{a}(O(T^{n_k}x^*)\, O(x^*))))$$

Corollary 2. 9 (X, q) *a quasi–cone metric space and* $T: X \to X$ *a* α*-admissible and* μ*-subadmissible such that* $\alpha(x, y)d(Tx, Ty) \leq \mu(x, y)\psi(M(x, y))$,*where* $\psi: P \to P$ *is a continuous function that satisfies the conditions* $\psi(0) = 0, \forall t \in P, \psi(t) < t$ *and* $\forall t_1, t_2 \in P$, for $t_1 < t_2$ implies $\psi(t_1) < \psi(t_2)$ and

1. $\alpha: X \times X \to [0, +\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1, \mu(T^n x_0, T^m x_0) \leq 1$ for every $n, m \in \mathbb{N}$.

2. If $\{T^n x\}$ is a sequence such that $\alpha(T^{n+1}x_0, T^n x_0) \geq 1, \mu(T^{n+1}x_0, T^n x_0) \leq 1$ for all $n \in N$ and $T^n x \to x^*$ as $n \to +\infty$, then there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$, such that $\alpha(T^{n_k}x, x^*) \geq 1, \alpha(x^*, T^{n_k}x) \geq 1, \mu(T^{n_k}x, x^*) \leq 1, \mu(x^*, T^{n_k}x) \leq 1$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Taking $h(s,t) = st$ and $F(x,y) = xy$ in Theorem 2. 7, the corollary is true.

Corollary 2. 10 *Let (X,q) a quasi–cone metric space and and $T: X \to X$ a α-admissible and such that $\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y))$, $\varphi, \psi: P \to P$ satisfies the conditions $\psi(0) = 0, \forall t \in P, \psi(t) < t$ and ψ is semi–lower continuous and*

1. $\alpha: X \times X \to [0,\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$for every $n, m \in \mathbb{N}$.

2. If $\{T^n x\}$ is a sequence such that $\alpha(T^{n+1}x_0, T^n x_0) \geq 1$for all $n \in N$ and $T^n x \to x^*$ as $n \to \infty$, then there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$ such that

$\alpha(T^{n_k}x, x^*) \geq 1, \alpha(x^*, T^{n_k}x) \geq 1$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Taking $h(s,t) = st$ and $F(x,y) = xy$ and $\mu(x,y) = 1$ in Theorem 2. 7, the corollary is proved.

Corollary 2. 11 *Let (X,q) a quasi-cone metric space and $T: X \to X$such that $d(Tx,Ty) \leq \psi(M(x,y))$, $\psi: P \to P$ satisfies the conditions $\psi(0) = 0, \forall t \in P, \psi(t) < t$ and ψ is increasing.*

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Taking $h(s,t) = st$ and $F(x,y) = xy$ and $\alpha(x,y) = 1, \mu(x,y) = 1$ in Theorem 2. 7, the corollary is proved.

3. Examples

Example 3. 1 *Let $X = [0,1], E = \mathbb{R}^2$, and P a normal cone with constant of normality K, $P = \{(x,y) \in \mathbb{R}^2, x,y \geq 0\}$. Determine $q: X \times X \to P$,*

$$q(x,y) = \begin{cases} (\frac{y}{3}, y), & x < y \\ (0,0), & x = y \\ (x, 3x), & x > y \end{cases}$$

is a quasi–cone metric and (X,q) is quasi-cone metric space.

Taking $h(x,y) = xy$ and $\mathcal{F}(x,y) = xy$ we have that the pair $(\mathcal{F}, h)$ is an cone upper class of type I.

Let $T: X \to X, T(x) = x^2$ be a continuous function, $\psi: P \to P, \psi(x,y) = (\frac{x}{3}, \frac{y}{3})$ be a comparison function and $\alpha: X \times X \to [0,\infty), \alpha(x,y) = \begin{cases} \frac{1}{2}\max(x,y), x \neq y \\ a, x = y \end{cases}$, where $a \geq 1$, $\mu: X \times X \to [0,\infty), \mu(x,y) = \begin{cases} 1, & x \neq y \\ b, & x = y \end{cases}$, where $b < 1$. The function T satisfies the condition of Theorem 2. 7.

Indeed, taking $x_0 = 0$ it is true that $\alpha(T^n x_0, T^m x_0) = a \geq 1$ and $\mu(T^n x_0, T^m x_0) = b < 1$.

Moreover, T satisfies the nonlinear contraction of Theorem 2. 7.

Case 1. $x = y$

This case is trivial because $q(Tx, Ty) = 0$.

Case 2. $x = y$

In this case

$$q(Tx,Ty) = q(x^2,y^2) = (\frac{y^2}{3}, y), \alpha(x,y) = \frac{y}{3}$$
$$\mu(x,y) = 1, h(\alpha(x,y), q(Tx,Ty)) = (\frac{y^3}{3}, y^3)$$
$$\delta(O(x) \cup O(y)) = \max\{q(x,y), q(x,Tx), q(y,Ty), q(x,T^i x), q(y,T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$
$$\delta(O(x) \cup O(y)) = (y, 3y), \psi(\delta(O(x) \cup O(y)) = (\frac{y}{3}, y)$$

Consequently, the nonlinear contraction of Theorem 2.7 is taken.

$$h(\alpha(x,y), q(Tx,Ty)) \leq \mathcal{F}(\mu(x,y), \psi(\delta(O(x) \cup O(y)))).$$

Case 3. $x > y$

$$q(Tx,Ty) = q(x^2,y^2) = (x^2, 3x^2), \alpha(x,y) = \frac{x}{3},$$

$$\mu(x,y) = 1, h(\alpha(x,y), q(Tx,Ty)) = (\frac{x^3}{3}, x^3)$$

$$\delta(O(x) \cup O(y)) = \max\{q(x,y), q(x,Tx), q(y,Ty), q(x,T^i x), q(y,T^j y),$$

$$q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (x, 3x), \psi(\delta(O(x) \cup O(y)) = (\frac{x}{3}, x)$$

As a result, the nonlinear contraction of Theorem 2.7 is completed.

$$h(\alpha(x,y), q(Tx,Ty)) \leq \mathcal{F}(\mu(x,y), \psi(\delta(O(x) \cup O(y)))).$$

So the function T has fixed points. The points $x = 0$ and $x = 1$ are the fixed points of T.

Example 3. 2 *Let* $X = [0,1], E = \mathbb{R}^2$, *and* P *be a normal cone,* $P = \{(x,y) \in \mathbb{R}^2, x, y \geq 0\}$. *Determine* $q: X \times X \to P$, *such that*

$$q(x,y) = \begin{cases} (\frac{y}{2}, y), & x < y \\ (0,0), & x = y \\ (x, 2x) & x > y \end{cases}$$

is a quasi–cone metric and (X, q) is quasi–cone metric space.

Let

$$T: X \to X, T(x) = \begin{cases} \frac{x^4}{3}, & 0 \leq x < \frac{1}{9} \\ \frac{1}{9}, & \frac{1}{9} \leq x \leq 1 \end{cases}$$

be a non–continuous function, $\psi: P \to P, \psi(x,y) = (\frac{x}{2}, \frac{y}{2})$ be a comparison function and $\alpha: X \times X \to [0, \infty)$,

$$\alpha(x,y) = \begin{cases} 0, & (x,y) \in [0, \frac{1}{9}) \times [\frac{1}{9}, 1] \cup [\frac{1}{9}, 1] \times [0, \frac{1}{9}) \\ b, & x = y \\ \frac{1}{3}\max\{x,y\}, & \text{otherwise} \end{cases}$$

where $b \geq 1$, $\mu: X \times X \to [0, \infty)$,

$$\mu(x,y) = \begin{cases} \frac{1}{2}, & (x,y) \in [0, \frac{1}{9}) \times [\frac{1}{9}, 1] \cup [\frac{1}{9}, 1] \times [0, \frac{1}{9}) \\ c, & x = y \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

where $0 \leq c < 1$.

The function satisfies the conditions of Theorem 2. 8.

Indeed, taking $x_0 = 0$ the following inequalities are true $\alpha(T^n x_0, T^m x_0) \geq, \mu(T^n x_0, T^m x_0) \leq 1$.

Also $T^n 0 \to 0$ when $n \to +\infty$, so $\alpha(T^n 0,0) \geq 1$, $\mu(T^n 0,0) \leq 1$.

Below is shown that the function T satisfies the nonlinear contraction of Theorem 2. 8.

Consequently, it needs to prove that for every $(x, y) \in X \times X$, the inequality

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\alpha(x,y))^i)q(Tx,Ty) \leq \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

holds.

Case 1. $x = y$. This case is trivial because $q(Tx, Ty) = 0$.

Case 2. 1. $x, y \in [0, \frac{1}{9}), x < y$. In this case $Tx = \frac{x^4}{3}, Ty = \frac{y^4}{3}, \alpha(x,y) = \frac{y}{3}, \mu(x,y) = y$,

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\frac{y}{3})^i)q(\frac{x^4}{3},\frac{y^4}{3}) = \frac{1}{n+1}\frac{1-(\frac{y}{3})^n}{1-\frac{y}{3}}(\frac{y^4}{6},\frac{y^4}{3})$$

$$\leq \frac{1}{n+1}\frac{3}{3-y}(\frac{y^4}{6},\frac{y^4}{3}) \leq \frac{3}{4}(\frac{y^4}{6},\frac{y^4}{3}) = (\frac{y^4}{8},\frac{y^4}{4}) \leq y(\frac{y}{2},y)$$

$$= \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

$$\delta(O(x) \cup O(y)) = \max\{q(x,y), q(x,Tx), q(y,Ty), q(x,T^i x), q(y,T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (y, 2y), \psi(\delta(O(x) \cup O(y)) = (\frac{y}{2}, y)$$

Case 2. 2. $x, y \in [0, \frac{1}{9}), x > y$ In this case $Tx = \frac{x^4}{3}, Ty = \frac{y^4}{3}, \alpha(x,y) = \frac{x}{3}, \mu(x,y) = x$, .

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\frac{x}{3})^i)q(\frac{x^4}{3},\frac{y^4}{3}) = \frac{1}{n+1}\frac{1-(\frac{x}{3})^n}{1-\frac{x}{3}}(\frac{x^4}{3},\frac{2y^4}{3}) \leq \frac{1}{n+1}\cdot\frac{3}{3-x}(\frac{x^4}{3},\frac{2y^4}{3}) \leq \frac{3}{4}(\frac{x^4}{3},\frac{2y^4}{3})$$

$= (\frac{x^4}{4}, \frac{y^4}{2}) \leq (\frac{x}{2}, y) = \mu(x,y)\psi(\delta(O(x) \cup O(y))\ \delta(O(x) \cup O(y))$
$= \max\{q(x,y), q(x,Tx), q(y,Ty), q(x,T^i x), q(y,T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$

$$\delta(O(x) \cup O(y)) = (x, 2y), \qquad \psi(\delta(O(x) \cup O(y)) = (\frac{x}{2}, y)$$

Case 3. 1. $x, y \in [\frac{1}{9}, 1], x < y$ or $x > y$. So, $Tx = \frac{1}{9}, Ty = \frac{1}{9}, q(Tx, Ty) = 0$, so

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\alpha(x,y))^i)q(Tx,Ty) \leq \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

Case 4. 1. $x \in [0, \frac{1}{9}), y \in [\frac{1}{9}, 1]$. Consequently, $Tx = \frac{x^4}{3}, Ty = \frac{1}{9}, \alpha(x,y) = 0, \mu(x,y) = \frac{1}{2}$, so

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\alpha(x,y))^i)q(Tx,Ty) \leq \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

Case 4. 2. $y \in [0, \frac{1}{9}), x \in [\frac{1}{9}, 1]$. In this case $Ty = \frac{y^4}{3}, Tx = \frac{1}{9}, q(Tx, Ty) = 0$, so

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\alpha(x,y))^i)q(Tx,Ty) \leq \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

So the function T has fixed points. The points $x = 0$ and $x = \frac{1}{9}$ are the fixed points of T.

4. Conclusion

Conclusion 4. 1 For $h(x,y) = xy$ and $\mathcal{F}(s,t) = st$ and $\mu(x,y) = 1$ in Theorem 2.7 and Theorem 2.8, there are taken the conditions of results of [17]. So, these results generalize the results of [17].
Conclusion 4. 2 Corollary 2.10 is a generalization of Theorem 3.4 of [1] and Theorem 14 of [2].

Conclusion 4. 3 Every result mentioned in corollaries above are true in case when it is replaced $M(x,y)$ by $\delta(O(x) \cup O(y))$, which are generalizations of results in [9].

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