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# Some properties of fractal tilings

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*Abstract*—**In 1975 Mandelbrot was first to introduce the concept of Fractals. A fractal is a complex geometric shape that continues to display self-similarity when viewed on all scales. The object need not exhibit exactly the same structure at all scales, but the same "type" of structures must appear on all scales. This type of shapes are frequently found in nature, from coastlines, to mountains and trees. The whole universe is a fractal. Fractal Mathematics has many practical uses, for example in computer file compression system, in the architecture of networks, in financial markets etc. The mathematics that is available for studying fractal sets has developed enormously. In this paper we review the method of construction of fractal tiling by using a Pisot number base. This type of tiling are interesting because we can study topological properties by using algebraic features. We will describe the differences between two type of Tilings, one generated by a Pisot number which has a finiteness property (F) and the other by a Pisot number which does not satisfy (F). In the end we will study some properties of two unstudied tilings.**

***Keywords- pisot number, tiling, beta expansion, greedy algorithm***

## I. INTRODUCTION

Let $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ be a polynomial where $a_i \in \mathbb{Z}$. Denote by $\beta^{(1)}=\beta, \beta^{(2)},\dots,\beta^{(n)}$ its roots. We call $\beta^{(2)},\dots,\beta^{(n)}$ conjugates of β.

**Definition 1.1.** β>1 is called a Pisot number if it is an algebraic integer whose conjugates other than itself have modulus smaller than 1.

## II. BETA EXPANSION AND GREEDY ALGORITHM.

Let **β**>1 be a real number. We are interested about the digit expansion of a positive number $x$ in base β. Below we give the algorithm for $x$ to be expressed as $a_{N_0}\beta^{N_0} + a_{N_0-1}\beta^{N_0-1} + \cdots$

where the coefficients $a_i \in \mathbb{Z} \cap [0,\beta)$.

For each real number $x>0$ it exists a unique $N_0 \in \mathbb{Z}$ such that $\beta^{N_0} \le x < \beta^{N_0+1}$.It is understandable that if $x=\beta^{N_0}$ the algorithm ends. Otherwise we do the same thing but now using the number $x-\beta^{N_0}$ instead of $x$. Since we have that $x-\beta^{N_0} \le x$ then $N_0 \ge N_1$ where $N_1$ satisfies the following condition

$$\beta^{N_1} \le x-\beta^{N_0} < \beta^{N_1+1}.$$

We repeat again the above procedure now using the number $x-\beta^{N_0}-\beta^{N_1}$ to find the number $N_2$ where $N_2 \le N_1$ which satisfies

$$\beta^{N_2} \le x-\beta^{N_0} - \beta^{N_1} < \beta^{N_2+1}$$

We can continue like this by finding the numbers $N_0 \ge N_1 \ge N_2 \ge \cdots$. Now we group the same terms to get

$x - a_{N_0}\beta^{N_0} - a_{N_1}\beta^{N_1} - \cdots = 0$

It is clear that $a_{N_0} \in \mathbb{N}$. Now let us show that

$a_{N_0} < \beta$. We suppose that

$a_{N_0}\beta^{N_0} = \beta^{N_0} + \beta^{N_1} + \cdots + \beta^{N_k}$

where $N_0 = N_1 = \cdots = N_k$. Let $N_{k+1}$ be the first number different from $N_k$ which satisfies the following condition

$$\beta^{N_{k+1}} \le x - \beta^{N_0} - \beta^{N_1} - \cdots - \beta^{N_k} < \beta^{N_{k+1}+1}.$$

By supposing the contrary that $a_{N_0} \ge \beta$ then we have

$-a_{N_0}\beta^{N_0} \le -\beta^{N_0+1}$

and by adding $x$ to both sides we get

$x - a_{N_0}\beta^{N_0} \le x - \beta^{N_0+1} < 0.$

The last inequality is an absurdity because we have that $x - a_{N_0}\beta^{N_0} > 0$.

So if we expand $x$ by usings the above algorithm then for all $N \in \mathbb{N}$

$0 \le x - \sum_{i=-N_0}^{N} a_{-i}\beta^{-i} < \beta^{-N}$ (G).

We call (G) Greedy condition and the algorithm we used Greedy -Algorithm.

Now let $T_\beta$: [0,1)→[0,1) be a map such that $T(x)=\{\beta x\}=\beta x-[\beta x]$, where $[\beta x]$ is the floor function.

It can be shown that β-expansion of a number in [0,1) by using the Greedy-Algorithm is the same as the expansion by using the map T.

Let $x = \sum_{i=-k}^{\infty} a_{-i}\beta^{-i}$ be a Greedy expansion. From now on we will denote this type of expansion as $x = a_k a_{k-1} \cdots a_0 . a_{-1} a_{-2} \dots$. The part denoted by $.a_{-1}a_{-2}\cdots$ is called the fractional part while the other part $a_k a_{k-1} \cdots a_0$ is called the integer part of $x$. We say this expansion is 'eventually periodic' if there exist a positive integer L such that $a_{-N} = a_{-N-L}$ for sufficiently large N and if the period start from $a_{-1}$ it is said to be 'purely periodic' . Finally if there exist $M$ such that $a_{-N} = 0$ for all $N \ge M$, the greedy expansion is said to be 'finite'.

It is worth mentioning that β expansion, where β is a Pisot number, has analogous properties with the usual decimal or binary expansion.

If $x = \sum_{N_0} a_{-i}\beta^{-i} = a_{-N_0}a_{-N_0-1}...a_{-M}[a_{-M-1}...a_{-M-L}]$, where $[a_{-M-1}...a_{-M-L}]$ is the periodic part,define function $M(x)$ and $L(x)$ respectively by the values of $M$ and $L$. In other words $M(x)$ is the last index of the non periodic part of $x$ and $L(x)$ is the length of the period of $x$.

Now let us expand $1-\frac{[\beta]}{\beta} = \sum_{i=2}^{\infty} c_{-i}\beta^{-i}$ by using Greedy-Algorithm. So formally we have

$1 = \sum_{i=1}^{\infty} c_{-i}\beta^{-i} = .c_{-1}c_{-2}\dots$, where $c_{-1} = [\beta]$.

This expansion is called the *characteristic sequence* and we say $d(1,\beta) = .c_{-1}c_{-2}...$ is the β expansion of number 1. It is understandable that the coefficients $c_{-2}, c_{-3}$, are produced by the map $T_\beta$. In general we use a modified expansion of number 1, which we denote by $d*(1,\beta)$.

Concretely

$$d*(1,\beta) = \begin{cases} d(1,\beta) & \text{if } d(1,\beta) \text{ is infinite} \\ (c_{-1}c_{-2}...c_{-M}-1)(c_{-1}c_{-2}...c_{-M}-1)... & \text{if } d(1,\beta) = c_{-1}c_{-2}...c_{-M} \end{cases}$$

## III. TILING CONSTRUCTION OF THE SPACE $\mathbb{R}^{n-1}$

Let $\mathcal{A}$={0,1,2,...[β]}. be a finite set. Consider the following set $\mathcal{A}^Z = \{ x = (x_i)_{i\in\mathbb{Z}} = ...x_2x_1x_0.x_{-1}x_{-2}... \mid x_i \in \mathcal{A} \}$.

We call the elements of $\mathcal{A}^Z$ words with symbols from $\mathcal{A}$.

Now let us take the shift operator $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that $(\sigma(x))_i = x_{i+1}$. So the shift operator moves every symbol to the left. We denote by $a \oplus b$ the concatenation of two words $a$ and $b$. We say the word $b$ serves as a tail for the word $\omega$ if it exist an non empty word $a$ such that $\omega = a \oplus b$. In the set of words with symbols from $\mathcal{A}$ we define an order relation, denoted $\preceq$, which is known as Parry order relation or lexicografical order.

**Definition3.1:** Let $a = a_m a_{m-1}..$ and $b = b_m b_{m-1}...$ be words in $\mathcal{A}^Z$. We say the word $a$ is *lexicografically smaller* than the word $b$ and denote $a \prec_{lex} b$ if from left to right the first index $s$ such that $a_i \neq b_i$ satisfies the condition $a_s < b_s$.

**Definition 3.2.** We say that the word $a = a_m a_{m-1}\dots$ Is *much smaller lexicographically* than the word $b = b_m b_{m-1}...$ and denote $a \underset{lex}{\ll} b$ if and only if for all $n \in \mathbb{Z}$ we have $a \underset{lex}{\ll} \sigma^n(b)$.

Where $\sigma^n(b) = \sigma(\sigma(\sigma...(\sigma(b))$ is the composition $n$ times of $\sigma$.

**Theorem 3.1. [5]** Let $x = a_m a_{m-1}...$ be a right infinite or finite word in $\mathcal{A}^{\mathbb{Z}}$, such that $x = a_m a_{m-1}... \underset{lex}{\ll} d*(1,\beta)$ then $x$ is a β greedy expansion.

So if $d(1,\beta) = .c_{-1}c_{-2}...c_{-M}$ ,is finite, we exlude from our consideration word whose tail is of the form $[c_{-1}c_{-2}...c_{-M+1}(c_{-M}-1)]$.Under this restriction, an infinite word generated by A is a greedy expansion on β if and only if such word is lexicographically less than the characteristic sequence at any starting point.

**Definition 3.3.** Let Fin(β) be a set of all finite greedy expansion in base β. We say β has property (F) if Fin(β)=**Z**[$\beta^{-1}$]$_{\geq 0}$

**Definition 3.4.** *A nonempty set of* $R^n$ *is called a tile if it coincides with the closure of its interior.*

If a finite set of tiles and their translations covers the space $R^n$ without overlapping then we say it forms a tiling. By without overlapping we mean that the translated tiles are reciprocally disjoint up to an n-dimensional set of Lebesgue measure 0.

W.P Thurston [1] was first to propose a method to construct a tiling of some Euclidean space by a Pisot unit with (F). Now we will describe the method of tiling construction of the space $\mathbb{R}^{n-1}$ by a Pisot unit of degree $n$.

Let $\mathbb{Q}(\beta)$ denote the smallest field containing both β and the rational numbers. Let $\beta = \beta^{(1)}, \beta^{(2)}, ...\beta^{(r_1)}, \beta^{(r_1+1)}, \overline{\beta^{(r_1+1)}}, ..., \beta^{(r_1+r_2)}, \overline{\beta^{(r_1+r_2)}}$ be respectively the real and complex conjugates of β. Define a map as following, $\phi : \mathbb{Q}(\beta) \to R^{r_1+2r_2-1}$ where,

$\phi(x) = (x^{(2)}, ..., x^{(r_1)}, \mathrm{Re}(x^{(r_1+1)}), \mathrm{Im}(x^{(r_1+1)}), ...\mathrm{Re}(x^{(r_1+r_2)}), \mathrm{Im}(x^{(r_1+r_2)}))$

. We denote by $x^{(j)}$ (*j=1,2,…n*) the corresponding conjugate of $x \in \mathbb{Q}(\beta)$ .Let $A = a_L a_{L-1}...a_M$, be a greedy expansion in base β .Put $\deg_\beta(A) = \deg(A) = L$ and $\mathrm{ord}_\beta(A) = \mathrm{ord}(A) = M$ Define $S_A$ to be the set of elements of Fin(β) whose greedy expansion has the suffix A. For the empty word $\lambda$ we define $S_\lambda = S = \{x \in Fin[\beta] | ord(x) \geq 0\}$. In other words we classify all elements of $\mathbb{Z}[\beta]_{\geq 0}$ by their fractional part and map them via $\phi$, to have a set $T_A = \overline{\phi(S_A)}$, where $\overline{\phi(S_A)}$ is the closure of the set $\phi(S_A)$ in the Euclidean topology of $R^{n-1}$ .Akiyama [2] has shown that $T_A$ is a tile and if β has property (F) then $\mathbb{R}^{n-1} = \bigcup_{\deg(A)=-1} T_A$. This results are corollary of the following propositions.

Let denote Inn($X$) the set of inner points of $X$.

**Proposition 3.1**. [2] *Let β be a Pisot unit with property (F). Then for each element* $x \in S$ *, we have* $\phi(x) \in Inn(T)$ .

**Proposition 3.2.** [2] Let β be a Pisot number of degree $n$. Then $\phi(\mathbb{Z}[\beta])_{>0}$ is dense in $\mathbb{R}^{n-1}$ .


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Pisot condition guarantees that $T_A$ is compact and the restriction to units is necessary to have a tiling by this construction.

A similar construction can be made even by a Pisot unit which does not satisfy property (F) .

Consider the following condition:

(W): For any element x of Z[1/β]≥0 and any positive ε, there exist two elements y, z in Fin(β) with |z| < ε such that x = y – z.

It is likely that all Pisot unit satisfy (W)[8]. In [7] we find several properties of tiling generated by a Pisot unit which satisfy condition (W).Denote by $\mathcal{P}$ the set of all purely periodic expansions in $Z[\beta]_{>0}$. The following propositions

**Propositon 3.3.**

*a)* *$\mathcal{P}$ is a finite set*

*b)* *$\mathcal{P}$={0} is equivalent with (F).*

Let **Fr** be the set of all fractional part of $Z[\beta]_{>0}$.

**Proposition 3.4**. $R^{n-1} = \bigcup_{\omega \in Fr} T_\omega$

**Propositon3.5.** *There are exactly M(1) + L(1) tiles up to translation.*

**Proposition 3.6.** *(Inflation-Subdivision principle). Any tile is subdivided into arbitrary small affine images of the tiles.*

**Propositon3.7.** *Suppose that β is a Pisot unit and consider the tiling generated by β. Then the origin 0 belongs to $T_\omega$ for any ω ∈$\mathcal{P}$ and 0 is an inner point of $\bigcup_{\omega \in P} T_\omega$ .*

By the above propostion which are due to Akiyama [7] we notice the difference of tiling generated by a Pisot unit with property (F) and of tiling generated by a Pisot unit without property (F). In the second case the origin may no longer be an inner point of a single tile but of a collection of tiles which correspond to purely periodic expansions. If a point $x$ is an inner point of a single tile we call it exlusive inner point. We consider the following condition:

(Ex): There exist an exclusive inner point in $T_\lambda$

**Proposition 3.8. [7]** *Conditon (Ex) is equivalent to (W).*

**Propositon 3.9. [7]** *Take any element $x \in S_\lambda$ . The point $\phi(x)$ is an exlusive inner point if and only if for any $K_0 \in N$ there exist $K \geq K_0$ that $\beta^K u + x \in S_\lambda$ for any $u \in \mathcal{P}$.*

**Theorem 3.2**. [7] *Let β be a Pisot unit with the property (W). Then $\mu(\delta(T_\omega)) = 0$ for $\omega \in Fr$ .*

Here $\mu = \mu_{n-1}$ is Lebesgue measure on $\mathbb{R}^{n-1}$ and $\delta(T_\omega) = T_\omega \setminus \mathrm{Inn}(T_\omega)$ is the boundary of $T_\omega$.

So if β is a Pisot unit which satisfy the property (W) then the tiling generated by β ,fully deserves to be called tiling.

Interesting results ,like the following Theorems, about this type of tilings we can find also in [9].

**Theorem 3.3.** [9] *Each tile corresponding to a Pisot unit β is arcwise connected if $d_\beta(1)$ terminates with 1.*

**Theorem 3.4.** [9]. *Let β be a Pisot unit of degree 3 or 4 defined by the monic polynomial $p(x) \in \mathbb{Z}[x]$. If degβ =3 or p(0)=1 then each tile is connected. If degβ=4 and p(0)=-1 then each tile is connected if and only if $a + c - 2[\beta] \neq 1$ for $p(x) = x^4 - ax^3 - bx^2 - cx - 1$.*

## IV. Examples

**Example 1.**

Now we will study some properties of tiling generated by pisot number $\beta \approx 1,61803$ which is a root of the quadratic equation $x^2 - x - 1 = 0$. The other root is $\beta' \approx -0.61803$ . We can find interesting examples of tilings generated by other Pisot Numbers in [3] and [4]. We have different expansion for number 1.

$1 = 0.11,\ \ 1 = 0.1011,\ \ 1 = 0.1010101(01)^\omega$.

But among them only the first one is admissible since it satisfies the 'lexicographically smaller' condition. Also we have that for a sequence of non negative integers $a_N, a_{N-1}...a_0, a_{-1}, a_{-2}...$ to be a greedy $\beta$ -expansion of a nonnegative number is that $a_i \in \{0,1\}$ with the condition:

(3) $a_i = 1 \Rightarrow a_{i+1} = 0$.

(3) is called the admissibility condition.

We notice that in this case M(1)=2 and L(1)=0, so there are only two kind of Tiles up to translation.

**Proposition 4.1.** *The pisot number $\beta$ satisfying the equation $x^2 - x - 1 = 0$ has property $(F)$.*

*Proof.*

In [6] it is shown that a Pisot number $\beta$ whose irreducible polynomial is of a form: $x^n - a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_0$ with $a_i \in Z_0$ and $a_i \geq a_{i-1} (i = 1, 2, \ldots n-1)$ then β has property (F).

Define as in (2) $\phi : Q(\beta) \to R$ where $\phi(x) = x^{(2)}$.

**Proposition 4.2.** *For each two tiles $T_1$ , $T_2$ the Lebesgue measure of their intersection equals zero.*

*Proof.*

In this case $T_A$ is a nonemtpy compact connected set in $\mathbb{R}$ , that is a closed interval.So we can conclude that the intersection of two different tiles is empty or a set with one point, therefore the Lebesgue measure of their intersection equals zero.

Let $p$ be a non negative integer and define $M_j(p)$ $(j=1,2,..,r_1+2r_2)$ by an upper bound of $|\sum_{i=0}^{p} a_{p-i}(\beta^{(j)})^i|$ where $\sum_{i=0}^{p} a_i\beta^{-i}$ runs through finite greedy expansion of length at most $p+1$. Let $M_j$ be an upper bound of $M_j(p)$ $(p=1,2...)$.

We may take $M_j=[\beta]/(1-|\beta^{(j)}|)$. In our case we may take $M_2=1/(1-(\beta')^2)=1.61803...$

Denote $D(x,r)=\{y\in R \,||\, y-x|<r\}$.

**Lemma 4.1.**
$T\subset D(0,M_2), T_{.10001}\subset D(\beta'^{-1}+\beta'^{-5},|\beta'|M_2)$.

*Proof.*

First we prove $T\subset D(O,M_2)$.

Let $x\in T=\overline{\varphi(S)}$.

*Case 1*: Let $x\in\phi(S)\Rightarrow x=\phi(y)$ where $y\in S\Rightarrow y=\sum_{i=0}^{n} a_i|\beta|^i$, where $a_i\in\{0,1\}$ and $a_i$ satisfy the admissibility condition. Then
$x=\sum_{i=0}^{n} a_i\beta'^i \Rightarrow |x|=|\sum_{i=0}^{n} a_i\beta'^i|$
$\le\sum_{i=0}^{n} a_i|\beta'|^i\le\sum_{i=0}^{n}|\beta'|^i<\sum_{i=0}^{\infty}(|\beta'|^2)^i=M_2$

*Case 2:* $x\in\overline{\phi(S)}$ and $x\notin\phi(S)$. Then for all $\varepsilon>0$ it exist $y\in\phi(S)$ such that $|x-y|<\varepsilon$.

So we have $|x|=|x-y+y|\le|x-y|+|y|<\varepsilon+M_2$. Since $y\in\phi(S)$ then $|y|<M_2$. $\lim_{\varepsilon\to0}|x|\le\lim_{\varepsilon\to0}\varepsilon+M_2$ $\Rightarrow|x|\le M_2$.

Now we show $T_{.10001}\subset D(\beta'^{-1}+\beta'^{-5},|\beta'|M_2)$.

We consider only the case $x\in\phi(S_{.10001})$ because the other case can be shown in the same way as above.

So we have $x=...0.10001=\sum_{i=1}^{N} a_i\beta'^i+\beta'^{-1}+\beta'^{-5}$. Then $|x-\beta'^{-1}+\beta'^{-5}|=|\sum_{i=1}^{N} a_i\beta'^i|\le\sum_{i=1}^{N} a_i|\beta'|^i$ $=$ $|\beta'|(\sum_{i=1}^{N} a_i|\beta'|^{i-1})\le|\beta'|M_2$.

Since $a_i\in\{0,1\}$ and satisfy the admissibility condition we have $\sum_{i=1}^{N} a_i|\beta'|^{i-1}\le M_2$.

In the same way we can prove similar inclusions that we will use in the following Proposition.

**Proposition 4.3.** *The central tile T does not intersect with any of the tiles of length 6,5,4,3.*

*Proof.*

We will show that $T$ does not intersect any of the tiles of length 4. Such tiles there are only three, precisely $T_{.0001}, T_{.0101}, T_{.1001}$. By using Lemma 4.1 we have that.

$T\cap T_{.1001}\subset D(0,M_2)\cap D(\beta'^{-1}+\beta'^{-4},\beta'M_2)=\varnothing$.

$T\cap T_{.0001}\subset D(0,M_2)\cap D(\beta'^{-4},M_2)=\varnothing$.

$T\cap T_{.0101}\subset D(0,M_2)\cap D(\beta'^{-2}+\beta'^{-4},M_2)=\varnothing$.

because by computer calculations, we have,

$\beta'^{-1}+\beta'^{-4}=1.6180...+6.8541...>2.6180...=M_2+|\beta'|M_2$

$\beta'^{-4}=6,8541...>3.23608...=2M_2$.

$\beta'^{-2}+\beta'^{-4}=9.47217...>3.23608...=2M_2$.

This is called the encircling method. In the same way we can show that T does not intersect any of the tiles of length 6,5,3.

But what happens with the tiles of length 2 an 1? By using the simple encircling method we cannot show that intersection of T with $T_{\cdot01}$ and of T with $T_{.1}$ is empty. We subdivide $T$ and $T_{\cdot01}$ as the union of three subtiles.

$T=T_{00}\cup T_{10}\cup T_1$.
$T_{\cdot01}=T_{\cdot00.01}\cup T_{10.01}\cup T_{1.01}$.
$T_{00}\cap T_{00.01}=T_{00}\cap T_{10.01}=T_{00}\cap T_{1.01}=\varnothing$.
$T_{00}\cap T_{00.01}=T_{10}\cap T_{10.01}=T_{00}\cap T_{1.01}=\varnothing$.

So if $T$ intersects $T_{.01}$ then $T_1$ must intersect any of $T_{00.01}$ or $T_{10.01}$ or $T_{1.01}$. Also we can show that $T_1\cap T_{00.01}=T_1\cap T_{1.01}=\varnothing$. But we cannot show that $T_1\cap T_{10.01}=\varnothing$. So finally we get that $T\cap T_{.01}=T_1\cap T_{10.01}$.

The above is a refined version of encircling method which we also call encircling method. Similarly by using the encircling method we get,

$T\cap T_{.1}=(T_{10}\cap T_{00.1})\cup(T_1\cap T_{00.1})$.

So we cannot show that the intersection of the central tile $T$ with respectively $T_{.1}$ and $T_{\cdot01}$ is empty.

**Theorem 4.1.** $-\beta'^{-1}\in T_{.01}\cap T_1$, $-1\in T_{10}\cap T_{.1}$

*Proof.*

Since $\beta^2-\beta-1=0\Rightarrow-\beta'=1-\beta'^2\Rightarrow$
$-1/\beta'=1/(1-\beta'^2)=1+\beta'^2+\beta'^3+\beta'^6+...\in T_1$ (*).

By multiplying both sides of equality (*) with $\beta'$ we take $-1=\beta'+\beta'^3+\beta'^5+...\in T_{10}$.

Also $\beta'^2-\beta'-1=0\Rightarrow1-\beta'^{-1}-\beta'^{-2}=0\Rightarrow$ $-\beta'^{-1}=\beta'^{-2}-1=\beta'^{-2}+\beta'+\beta'^3+\beta'^5+...\in T_{.01}$(**) So


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$-(\beta')^{-1} \in T_1 \cap T_{.01}$. By multiplying both sides of equality (**) with $\beta'$ we take $-1 \in T_{.1}$.

We can obtain the concrete shape of the central tile $T_\lambda$ by computing the extremal values.

$$T_\lambda = \{\sum_{i=0}^{\infty} a_i (\beta')^i \,|\, a_{i+1}, a_i \underset{lex}{<} 1,1\} =$$

$$\left[\sum_{k=1}^{\infty} (\beta')^{2k-1}, \sum_{k=0}^{\infty} (\beta')^{2k}\right] = \left[\frac{\beta'}{1-(\beta')^2}, \frac{1}{1-(\beta')^2}\right] = [-1, \beta]$$

So the central tile is the segment [-1,β]. At the left the central tile intersect with the tille $T_{.1}$ and at the right the central tile intersect with the tile $T_{.01}$.

**Example 2.**

Now will study some properties of the 'tiling' generated by the Pisot number β =2,8978 which is the root of the polynomial $p(x) = x^4 - x^3 - 4x^2 - 4x - 1$.

The other roots are $\beta^{(2)} = -0.36892$, $\beta^{(3)} = -0.76444 + 0.59249i$, $\beta^{(4)} = -0.76444 - 0.59249i$

By performing greedy algorithm to the number $1 - \frac{[\beta]}{\beta}$,we get that the characteristic sequence is

1=.221[2010211011],

Where [2010211011] is the periodic part.

**Proposition 4.3**. *$\beta$ does not satisfy property (F).*

*Proof.*

One can show that the set of purely periodic elements $\mathcal{P}$={0,.1111…}. From Propositon 3.3 we get that β does not satisfy property (F).

Define as in (2) the map $\phi : \mathbb{Q}(\beta) \to \mathbb{R}^3$ ,where $\phi(x) = (x^{(2)}, \mathrm{Re}(x^{(3)}), \mathrm{Im}\, x^{(3)})$ . We recall that if $x$ is an algebraic integer and $p(x)$ is its minimal polynomial of degree *n* then $\{1, \beta, \beta^2, ..., \beta^{n-1}\}$ is a base of the vectorial space $\mathbb{Q}(\beta)$ over the field $\mathbb{Q}$ i.e for every $x \in \mathbb{Q}(\beta)$ there exist $a_0, a_1, ..., a_{n-1} \in \mathbb{Q}$ such that $x = a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{n-1}\beta^{n-1}$. So in our case if $x = a_0 + a_1\beta + a_2\beta^2 + a_3\beta^3$ then

$$x^{(2)} = a_0 + a_1\beta^{(2)} + a_2(\beta^{(2)})^2 + a_3(\beta^{(2)})^3$$

$$x^{(3)} = a_0 + a_1\beta^{(3)} + a_2(\beta^{(3)})^2 + a_3(\beta^{(3)})^3 .$$

**Proposition 4.4**: *Each tile is disconnected.*

*Proof:*

Since we have the polynomial $p(x) = x^4 - x^3 - 4x^2 - 4x - 1$ and [β ]=2, then $a + c - 2[\beta] = 1 + 4 - 2*2 = 1$ . By using Theorem 3.2 we conclude that not every tile is connected. Also in [9] is stated that if it exist a disconnected tile then all the tiles are disconnected.

In this case we use the word 'tiling' in parentheses because till now we have not proved that the intersection of any two tiles has Lesbegue measure equal to zero. It is obvious that if β satisfies property (W), by using Theorem 10 we are sure that the intersection of any two tiles has Lesbegue measure equal to zero. But until now we could not prove that β satisfy property (W).

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